Suggested Solution 5

1. We summarize the properties of the Lebesgue measure on $\mathbb{R}^n$. For $E \subset \mathbb{R}^n$, let

$$\mathcal{L}^n(E) = \inf \left\{ \sum_k |C_k| : \ E \subset \bigcup_k C_k, C_k \text{ closed cubes} \right\} .$$

We have

(a) $\mathcal{L}^n(C) = 1$ for every unit cube C (open or closed).

(b) $\mathcal{L}^n$ is a σ-finite Borel measure.

(c) $\mathcal{L}^n$ is finite on bounded sets.

(d) For every measurable E,

$$\mathcal{L}^n(E) = \inf \{ \mathcal{L}^n(G) : \ E \subset G, \ G \text{ open} \} ;$$

$$\mathcal{L}^n(E) = \sup \{ \mathcal{L}^n(K) : \ K \subset E, \ K \text{ compact} \} \ .$$

(e) Let T be a linear transformation from $\mathbb{R}^n$ to itself. For each measurable E, $T(E)$ is also measurable and there is some constant C_T such that

$$\mathcal{L}^n(T(E)) = C_T \mathcal{L}^n(E) \ .$$

(a)-(d) were covered in previous exercises. Prove (e).

Solution. You are referred Proposition 3.2 in my notes or Theorem 2.20 (e) in [R].

2. Let Φ be a Lipschitz continuous map on $\mathbb{R}^n$ to $\mathbb{R}^n$, that is, for some $L > 0$,

$$|\Phi(x) - \Phi(y)| \le L|x - y| \ , \quad \forall x, y \in \mathbb{R}^n \ .$$

Show that $\Phi(E)$ is measurable if E is (Lebesgue) measurable.

Solution. Assume that E is compact first. As the image of a compact set under a continuous map is again compact and so is Borel, we see that $\mathcal{L}^n(E)$ is also compact, hence measurable. Next, let E be a bounded measurable set. By inner regularity we can find a set $F \subset E$ which is the countable union of compact sets satisfying $\mathcal{L}^n(E \setminus F) = 0$.

Hence the set $N = E \setminus F$ is null and $\Phi(E) = \Phi(F) \cup \Phi(N)$. We have $\Phi(F) = \bigcup_j \Phi(K_j)$ where K_j are compact, so $\Phi(F)$ is Borel (hence measurable). Therefore, things boil down to show that the image of a null set under a Lipschitz map is a null set. To show that $\Phi(U)$ is of $\mathcal{L}^n$-measure zero if U is, by the propositions in lecture notes Ch3, we have

$$\begin{aligned}
\mathcal{L}^n(\Phi(U)) &\ll \mathcal{H}^n(\Phi(U)) \quad (Proposition 3.10)\\
&\ll \mathcal{H}^n(U) \qquad\quad (Proposition 3.11)\\
&\ll \mathcal{L}^n(U) \qquad\quad (Proposition 3.10)\\
&= 0.
\end{aligned}$$

where $\mathcal{H}^n$ is the n-dimensional Hausdorff measure and " $f \ll g$ " means for given $f, g \geq 0$, there exists a constant $C \geq 0$ such that $f \leq Cg$.

Finally, we can write a measurable set as the countable union of bounded, measurable sets.

3. This problem is related to the σ-finiteness condition in Proposition 2.10. Define the distance between points (x_1, y_1) and (x_2, y_2) in the plane to be

$$|y_1 - y_2| \quad \text{if } x_1 = x_2, \qquad 1 + |y_1 - y_2| \quad \text{if } x_1 \neq x_2.$$

Show that this is indeed a metric, and that the resulting metric space X is locally compact. If $f \in C_c(X)$, let $x_1, \ldots, x_n$ be those values of x for which $f(x, y) \neq 0$ for at least one y (there are only finitely many such x!), and define

$$\Lambda f = \sum_{j=1}^{n} \int_{-\infty}^{\infty} f(x_j, y)\, dy.$$

Let μ be the measure associated with this Λ by the representation theorem. If E is the x-axis, show that $\mu(E) = \infty$ although $\mu(K) = 0$ for every compact $K \subset E$.

Solution. Write $p_i = (x_i, y_i)$, $i = 1, 2$. Denote

$$d(p_1, p_2) = \begin{cases} |y_1 - y_2|, & x_1 = x_2, \\ 1 + |y_1 - y_2|, & x_1 \neq x_2. \end{cases}$$

We prove that d is a metric.

- $d(p_1, p_2) \geq 0$ and $d(p_1, p_2) = 0$ if and only if $p_1 = p_2$.
- $d(p_1, p_2) = d(p_2, p_1)$.

- $d(p_1, p_2) \le d(p_1, p_3) + d(p_3, p_2)$ holds because $|y_1 - y_2| \le |y_1 - y_3| + |y_3 - y_2|$.

Now we claim that (X, τ) is a locally compact Hausdorff space. Let τ_1 be the discrete topology on $\mathbb{R}$, so every singleton $\{x\}$ is an open set. Then every point $x \in \mathbb{R}$ has the compact set $\{x\}$ as a neighborhood, so that $(\mathbb{R}, \tau_1)$ is a locally compact Hausdorff space. Note that $(X, \tau) = (\mathbb{R}, \tau_1) \times (\mathbb{R}, \tau_2)$, where τ_2 is the usual topology of $\mathbb{R}$. The claim follows.

If K is compact in X, the first projection $\mathrm{pr}_1(K)$ is compact in $(\mathbb{R}, \tau_1)$. Hence it is a finite set. Therefore K is a finite union

$$\{x_1\} \times K_1 \cup \cdots \cup \{x_n\} \times K_n,$$

where each K_i, $i = 1, 2, \ldots, n$, is a compact set in $(\mathbb{R}, \tau_2)$.

If $f : X \to \mathbb{R}$ has compact support, then $\operatorname{spt} f \subset \{x_1, \ldots, x_n\} \times \mathbb{R}$. Thus,

$$\Lambda f = \sum_{j=1}^{n} \int_{-\infty}^{\infty} f(x_j, y)\, dy$$

defines a positive linear functional on $C_c(X)$.

By the proof of Riesz's representation theorem, the measure μ defined by the equalities

$$\begin{aligned} \mu(V) &= \sup_{K \subset V \text{ compact}} \mu(K) = \sup_{f \prec V} \Lambda f, \\ \mu(E) &= \inf_{V \supset E \text{ open}} \mu(V) \end{aligned}$$

is a representing measure for Λ. Using the second equality with the Lebesgue measure m on $\mathbb{R}$, we observe that

$$\mu(\{x\} \times K) = m(K).$$

Thus μ is characterized by the identity

$$\mu(\{x\} \times [a, b]) = b - a, \qquad x \in \mathbb{R}.$$

Let V be an open set containing $\mathbb{R} \times \{0\}$. Then for $x \in \mathbb{R}$, $(x, 0) \in V$, so that there exists an $\varepsilon_x > 0$ with

$$\{x\} \times [-\varepsilon_x, \varepsilon_x] \subset V.$$

This implies that there must be an n with uncountably many $\varepsilon_x \ge 1/n$. (Otherwise, $\varepsilon_x \ge 1/n$ for at most countably many x, contradicting the fact that $\mathbb{R}$ is uncountable.)

Let

$$K_x = \{x\} \times \left[-\frac{\varepsilon_x}{2}, \frac{\varepsilon_x}{2}\right], \quad \varepsilon_x \geq \frac{1}{n}.$$

For $K = \bigcup_{j=1}^{m} K_{x_j}$, we have $\mu(K) \geq \frac{m}{n}$. Hence, if $V \supset \mathbb{R} \times \{0\}$ is open, then $\mu(V) \geq \sup_{m\in\mathbb{N}} \frac{m}{n} = \infty$. This implies $\mu(\mathbb{R} \times \{0\}) = \infty$.

Now if K is a compact subset of $\mathbb{R} \times \{0\}$, then $K = \{x_1, \ldots, x_n\} \times \{0\}$, which implies $\mu(K) = 0$.

Therefore for $E = \mathbb{R} \times \{0\}$, $\mu(E) = \infty$ while $\sup_{K\subset E \text{ compact}} \mu(K) = 0$. This means that μ is not inner regular.

4. Let μ be a Borel measure on $\mathbb{R}^n$ such that $\mu(K) < \infty$ for all compact K. Show that μ is the restriction of some Riesz measure on $\mathcal{B}$.

   Solution. First prove that if μ is a Borel measure and μ_Λ is a Riesz measure on $\mathbb{R}^n$ such that $\mu(G) = \mu_\Lambda(G)$ for all open sets G, then μ coincides with μ_Λ on $\mathcal{B}$.

   Let $E \in \mathcal{B}$. For $\varepsilon > 0$, by Proposition 2.10, there exists an open set G and a closed set F with $F \subset E \subset G$ such that $\mu_\Lambda(G \setminus F) < \varepsilon$. Since G and $G \setminus F$ are open, μ and μ_Λ coincide on them, and one has

$$\begin{aligned}\mu_\Lambda(E) &= \mu_\Lambda(G) - \mu_\Lambda(G \setminus E) \geq \mu_\Lambda(G) - \mu_\Lambda(G \setminus F) = \mu(G) - \mu(G \setminus F) \\ &\geq \mu(E) - \varepsilon.\end{aligned}$$

   By changing the position of μ_Λ and μ, one has

$$\mu(E) - \varepsilon \leq \mu_\Lambda(E) \leq \mu(E) + \varepsilon.$$

   Since this holds for any $\varepsilon > 0$, one has $\mu_\Lambda(E) = \mu(E)$.

   Now it suffices to show both measures coincide on open sets. For $f \in C_c$, define the linear functional by

$$\Lambda f = \int f d\mu \ .$$

   As μ is finite on compact sets, this is a well-defined and obviously a positive functional. By the representation theorem there is a Riesz measure μ_Λ such that

$$\int f d\mu = \int f \mu_\Lambda \ , \quad \forall f \in C^c(\mathbb{R}^n) \ .$$

For any open set G, we can find an ascending sequence of compact sets $\{K_n\}$ such that $G = \bigcup_n K_n$. Let f_n satisfy $K_n < f_n < G$ so that f_n increases to χ_G pointwisely. By Lebesgue monotone convergence we get

$$\mu(G) = \lim_{n\to\infty} \int_G f_n d\mu = \lim_{n\to\infty} \int_G f_n d\mu_\Lambda = \mu_\Lambda(G) \ .$$